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Instability of the wet X soap film

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Abstract

For idealized, infinitely thin ("dry") soap films, an X is unstable, while for very thick ("wet") soap films it is minimizing. We show that for soap films of relatively small but positive wetness, the X is unstable. Full stability diagrams for the constant liquid fraction case and the constant pressure case are generated. Analogous questions about other singularities remain controversial.

1 Introduction

In his famous experiments, J. A. F. Plateau [P] observed and recorded that pieces of soap film meet only in threes (at 120 degrees) along singular curves which meet only in fours (at about 109 degrees). J. Taylor [T] showed that for idealized, infinitely thin ("dry") soap films, Plateau's laws follow mathematically from a hypothesis of area minimization. The question is whether other kinds of singularities can be stabilized by wetting the film with a small amount of fluid that more realistically provides for small but positive thickness regions where films meet, called "Plateau borders". For the singularity formed by a cone over a cubical frame, experiments and very rough calculations by D. Weaire and R. Phelan [WP] suggested that a wet cone over a cubical wire frame is stable down to arbitrarily low liquid fractions, but recent computer simulations [B] provide strong evidence that the wet cube cone becomes unstable for liquid fractions below about 0.000274.

This paper considers the simpler "X" singularity of two planes crossing at some angle, and proves the slightly wet X unstable. To prove instability it suffices to consider the analogous, one-dimensional planar soap film connecting

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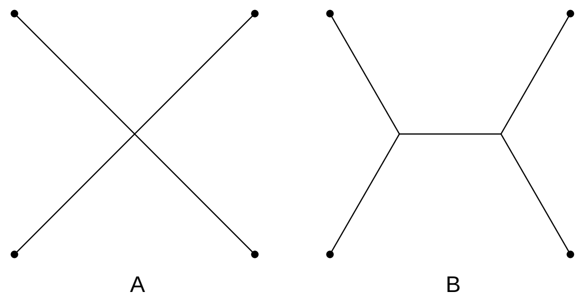


Figure 1: A dry planar X (left) is unstable to converting to a dry double Y (right). The dry double Y may also be at 90° from that pictured.

four vertices at the corners of a rectangle. Such a dry film in the form of an X (figure 1A) can reduce length to first order by converting to a double Y as in figure 1B, although it is stationary as a varifold [A] in the sense of forces being balanced. Dry planar films meet only in threes (at 120 degrees). For a wet film, there can be several equilibria, pictured in figure 2. The stability of these is the subject of this paper.

It should be mentioned that there is a qualitative difference between the 2-dimensional X in R^3 and the cone over a cubical wire frame (and the other six unstable soap film cones classified by Heppes, Lamarle, and Taylor ([H1], [H2], [L], [T]; see [M2,§13.9])). The latter have nonnegative first variation under deformations $f(x,t)$ of the ambient space which are Lipschitz in x and t, while the X does not (although it does under smooth deformations). Lipschitz variations (in contrast with smooth variations [A,§4]) are not well understood. In general they are not linear functions of an initial velocity and it is not even known whether nonnegative first variation is a local condition. Smooth minimal surfaces do have nonnegative Lipschitz first variation, although for very wiggly Lipschitz deformations, the first variation can be positive.

Section 2 finds the possible equilibrium configurations of wet films joining the corners of a rectangle. Section 3 decides stability of the equilibria in the case of fixed liquid area. Theorem 3.1 computes that for a small amount of fluid the X is unstable, and provides a complete description of the phenomenon. Consequently the 2-dimensional X in $\mathbf{R}^3$ is unstable for small amounts of fluid. Section 4 does likewise for fixed pressure. Sections 3 and 4 consider planar wet films of fixed topological type, for which existence and regularity come from [HM]. Section 5 gives existence and regularity for wet films in $\mathbf{R}^n$ of unrestricted topological type.

A Java applet showing all the equilibria is available at `http://www.susqu.edu/facstaff/b/brakke/wetx`. This web page also has

VRML files for 3D versions of figures 7 and 13.

After drafting this paper we discovered the paper of F. Bolton and D. Weaire [BW], which presents without proof a comprehensive description of the stability and evolution of the wet X for constant pressure under changing boundary conditions. Excellent surveys on foams are provided by Weaire and M. A. Fortes [WF] and Weaire and N. Rivier [WR].

2 Possible equilibria.

This section describes all types of equilibrium wet soap films bounded by the four corners of a rectangle.

The mathematical model we use for a wet soap film is a circle shrunk down around a set of fixed points (the rectangle corners). Theorem 2.1 states the existence of length minimizers, and shows them to be composed of straight line segments and circular arcs. It is a minor modification of Theorem 3.1 and §3.7 of [HM]. The same proof holds.

Theorem 2.1 *Let P be a finite set of points in the open unit disk $\{|x| < 1\} \subset \boldsymbol{R}^2$, let Q be the annulus $\{1/2 \leq |x| \leq 1\}$, and let $\mathcal{F}$ be the closure of the space of smooth embeddings of $\{|x| = 1/2\}$ into $\{|x| < 1\} - P$ which extend to smooth embeddings of Q into $\{|x| \leq 1\}$ that leave $\{|x| = 1\}$ fixed. Given $0 < A < area(convex\ hull(P))$, there is a length-minimizing curve $f \in \mathcal{F}$ enclosing a region of area A. It is an embedded curve of constant positive outward curvature except for the points of P and finitely many straight segments or isolated points where it has multiplicity two.*

Remark. Theorem 2.1 generalizes from $\{|x| \leq 1\}$ to other nice domains R containing P with $A < area(R)$. The curvature of f need not be positive, and f may touch or adhere to portions of the boundary of greater or equal curvature.

Definitions. The members of $\mathcal{F}$ will be referred to hereafter as *wet films*. Since "constant curvature" is a consequence of zero first variation, wet films satisfying the last sentence of Theorem 2.1 will be called *equilibria*. Equilibria are *stable* if no nearby admissible film has less length. A corner is *wet* if it is on the boundary of the interior area and the two film segments make a positive angle, otherwise it is *dry*. A film is *fat* if all corners are wet.

The following theorem classifies all types of equilibrium wet films on the corners of a rectangle.

Theorem 2.2 *Consider wet films with four corners fixed at $\{(\pm x, \pm y)\}$ of interior area $A < 4xy$, composed of straight segments of multiplicity two and circular arcs of equal positive outward curvature. Any such configuration has rectangular symmetry, and falls into one of four types:*

1. *An X, with four dry corners and no opposite sides touching (fig. 2A,B,C).*

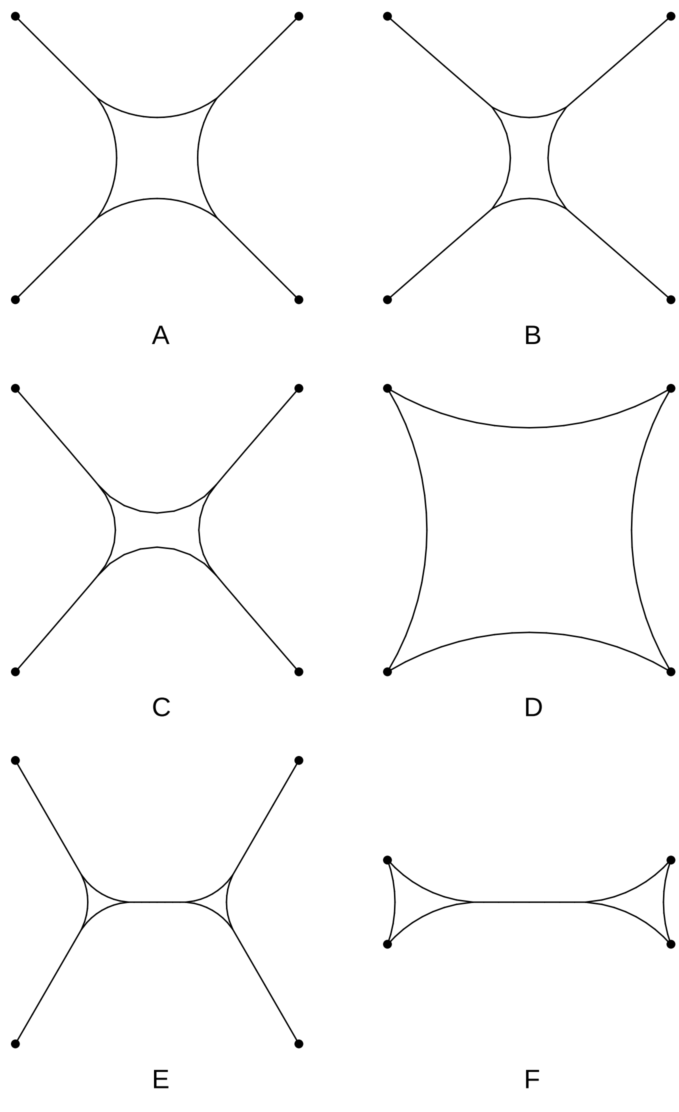


Figure 2: Possible equilibrium films enclosing four corners of a rectangle. Films A, B, and C will be referred to as X films, film D as a fat X, film E as a horizontal double Y, and film F as a fat horizontal double Y. Films E and F also have vertical versions. “Fat” refers to the liquid reaching the vertices with a positive angle.

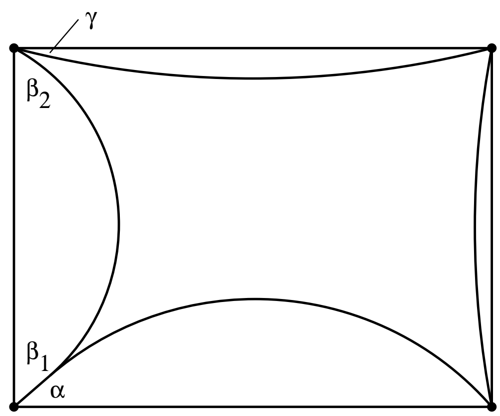


Figure 3: Angles in a hypothesized X film with one dry corner, the lower left.

2. *A unique fat* X*, with four wet corners and no opposite sides touching (fig. 2D).*
3. *A unique horizontal and/or unique vertical double* Y*, with four dry corners and a pair of opposite sides touching (fig. 2E).*
4. *A unique fat double* Y*, with four wet corners and a pair of opposite sides touching (fig. 2F).*

Proof. Topologically, an equilibrium must be X-like (opposite sides not touching) or Y-like (a pair of opposite sides touching). We have to prove symmetry, uniqueness as asserted, and the fact that either all four corners are wet or all four corners are dry.

X-like films. First we show that the corners of an X-like film are either all wet or all dry.

If exactly one corner is dry, as in figure 3, then $\alpha \leq \gamma$ and $\beta_1 \leq \beta_2$, and hence $\pi/2 = \alpha + \beta_1 \leq \gamma + \beta_2 < \pi/2$, a contradiction.

If exactly two diagonally opposite corners are dry, as in figure 4, then

$$\pi = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \leq \alpha_2 + \beta_2 + \gamma_2 + \delta_2 < \pi, \tag{1}$$

a contradiction.

If there are two adjacent dry corners, say P and Q, followed by wet corners R and S, then R will have to lie on arc DC of figure 5 and S will lie on arc AB. If one plots the successive vertices of a rectangle lying on these curves, the vertices follow a monotonic rectangular spiral. The only way for the spiral to close up into a rectangle is if all four corners are symmetrically placed, and hence all dry or all wet.

If there are three dry corners, suppose P and Q are dry corners followed by wet corner R. Then dry corner S is on a ray tangent to arc AB in figure 5,

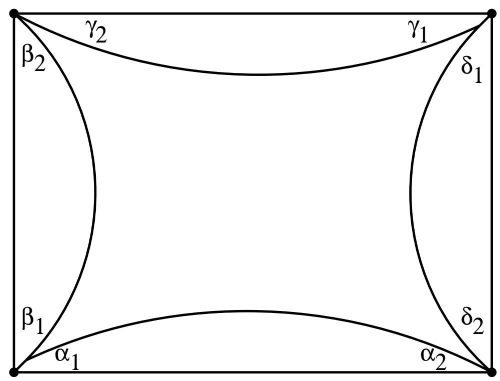


Figure 4: Angles in a hypothesized X film with opposite dry corners, lower left and upper right.

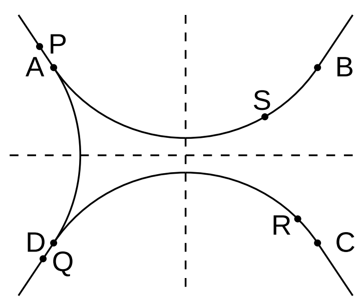


Figure 5: Corner locus for X film with at least two adjacent dry corners, taken to be on the straight segments at left. The other two corners must lie on the upper and lower curves extending to the right. Equal radius arcs continue as rays at the tangency points A,B,C,D.

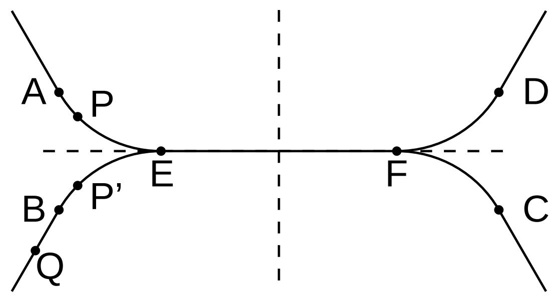


Figure 6: Corner locus for a Y-like film. One corner must lie on each of the four curves. Arcs continue as tangent rays at points A,B,C,D.

hence under the arc-ray curve AB. Hence the spiral rectangle will be even less able to close up than if S were on the curve.

Second, we consider the symmetry of X's with all dry or all wet corners. If all corners are dry, then the preceding paragraph shows the X is symmetric. If all corners are wet, then the corners are joined by circular arcs of equal curvature, hence the film is a symmetric fat X. The area is monotonic in the curvature, so the fat X is unique.

Y-like films. The four corners lie on the four curves shown in figure 6, each of which consists of an arc and a ray. In general, the arcs need not be $\pi/3$. For rectangle side PQ the ray EF points away from the perpendicular bisector of PQ. Since the corresponding statement holds for RS, the only way for the two halves of the film to match up is for EF to be the perpendicular bisector of PQ and RS, implying rectangular symmetry.

Having rectangular symmetry, it follows that all corners are wet or all are dry. If all the corners are dry, then the film is a double Y, and if all wet , it is a fat double Y. Q.E.D.

Remark on higher dimensions. For dry films on extremal curves in $\mathbf{R}^3$, F. Almgren ([A2, Thm. 22], described and pictured in [AT, p. 84]) provides the existence and regularity of a least-area sphere (shrunken down to volume $V = 0$) containing the boundary. The approach does not seem to generalize to wet films ($V > 0$) because of the lack of regularity results for the limit varifold.

3 Fixed liquid area.

We here catalog all the equilibria for fixed liquid area and find their stability. Instability can be proven by finding length decreasing perturbations, and stability is usually obvious from having the minimum length among equilibria.

However, there are some cases of stable equilibria that are only local minima, so some work is needed to prove these stable.

Theorem 3.1 *Consider wet films with four corners fixed at $(\pm x, \pm y)$, with prescribed area $4A$ where $0 < A < xy$. Define*

$$r_0 = \left[\frac{A}{\frac{\sqrt{3}}{2} - \frac{\pi}{4}} \right]^{1/2}. \tag{2}$$

The possible equilibrium films are, with reference to figure 7:

1. *A stable vertical double* Y *in regions B, D, and G:*

$$\frac{r_0}{2} \leq x < \sqrt{3}y - r_0. \tag{3}$$

2. *A stable horizontal double* Y *in regions B, C, and F:*

$$\frac{r_0}{2} \leq y < \sqrt{3}x - r_0. \tag{4}$$

3. *A stable horizontal fat double* Y *in region H: $y < r_0/2$ and*

$$\begin{aligned} \frac{x^3 + 3xy^2}{4y} &+ \frac{1}{4}\left[x^4 + 2x^2y^2 - 3y^4\right]^{1/2} \\ &- \frac{1}{2}\left(\arcsin \frac{2xy}{x^2+y^2} + \arcsin \frac{2y^2}{x^2+y^2}\right)\left[\frac{x^2+y^2}{2y}\right]^2 \geq A. \end{aligned} \tag{5}$$

4. *A stable vertical fat double* Y *in region I: $x < r_0/2$ and*

$$\begin{aligned} \frac{3x^2y + y^3}{4x} &+ \frac{1}{4}\left[y^4 + 2x^2y^2 - 3x^4\right]^{1/2} \\ &- \frac{1}{2}\left(\arcsin \frac{2x^2}{x^2+y^2} + \arcsin \frac{2xy}{x^2+y^2}\right)\left[\frac{x^2+y^2}{2x}\right]^2 \geq A. \end{aligned} \tag{6}$$

5. *An* X *in regions A, B, E, G, and F:*

$$2xy - \frac{\pi}{4}(x^2 + y^2) \geq A \tag{7}$$

and there is some $\pi/6 < t < \pi/3$ such that

$$A = \left(\frac{x \cos t - y \sin t}{\cos^2 t - \sin^2 t}\right)^2 \left(\sin 2t - \frac{\pi}{4}\right). \tag{8}$$

There are one stable X *in region A, two stable* X*'s and one unstable* X *in region E, one stable* X *and one unstable* X *in regions F and G, and one* *unstable* X *in region B. The curves PQ and PR are defined by* $\partial A/\partial t = 0$*, and indicate the merging of a stable* X *and an unstable* X *on one side of the curve into an unstable* X *on the curve. On rays TQ and SR* ($t = \pi/6$ *and* $t = \pi/3$*, respectively), opposite sides meet at one point, and this double* Y *is stable on segments TQ and SR, but unstable at Q and R and beyond.*

6. *A stable fat* X *for* (x, y) *in region J, the region of* $xy < A$ *not otherwise covered. (Technically, in this theorem, region J is bounded by the hyperbola* $xy = A$*, but that curve is not shown in figure 7 since nothing significant really happens there. It is not until region K is reached that changes occur, when the fat* X *develops bulging semicircles.)*

Proof. Theorem 2.1 provides the existence and regularity of minima. The minima must be among the possibilities listed in Theorem 2.2, so the main question is one of stability for (x, y) where multiple equilibria exist. The (x, y) domains for the various cases are easily computed by simple geometry to be those listed in this theorem. Using the rectangular symmetry, we need only work with one quadrant. The r_0 mentioned is the radius of curvature of the double Y for the given area. Figure 8 shows the transition between the double Y and the fat double Y, with the arcs tangent at S, so the equation for the dividing line between regions C and H in figure 7 is $y = r_0/2$. By symmetry, the dividing line between regions D and I is $x = r_0/2$.

Figure 9 shows the transition between the X and the double Y, along line RS in figure 7. The Plateau border is uniquely determined, and the point (x, y) lies on the ray tangent to the arcs, whence $y = \sqrt{3}x - r_0$. Likewise, the line TQ is $x = \sqrt{3}y - r_0$.

Figure 10 shows the transition between the fat X and the fat double Y. From $x^2 + (y - r)^2 = r^2$, one gets $r = (x^2 + y^2)/2y$, and the area is a trapezoid plus a triangle minus two circular sectors:

$$A = x(r + y)/2 + \frac{1}{2}y\sqrt{r^2 - y^2} - \frac{s}{2}r^2 - \frac{t}{2}r^2. \tag{9}$$

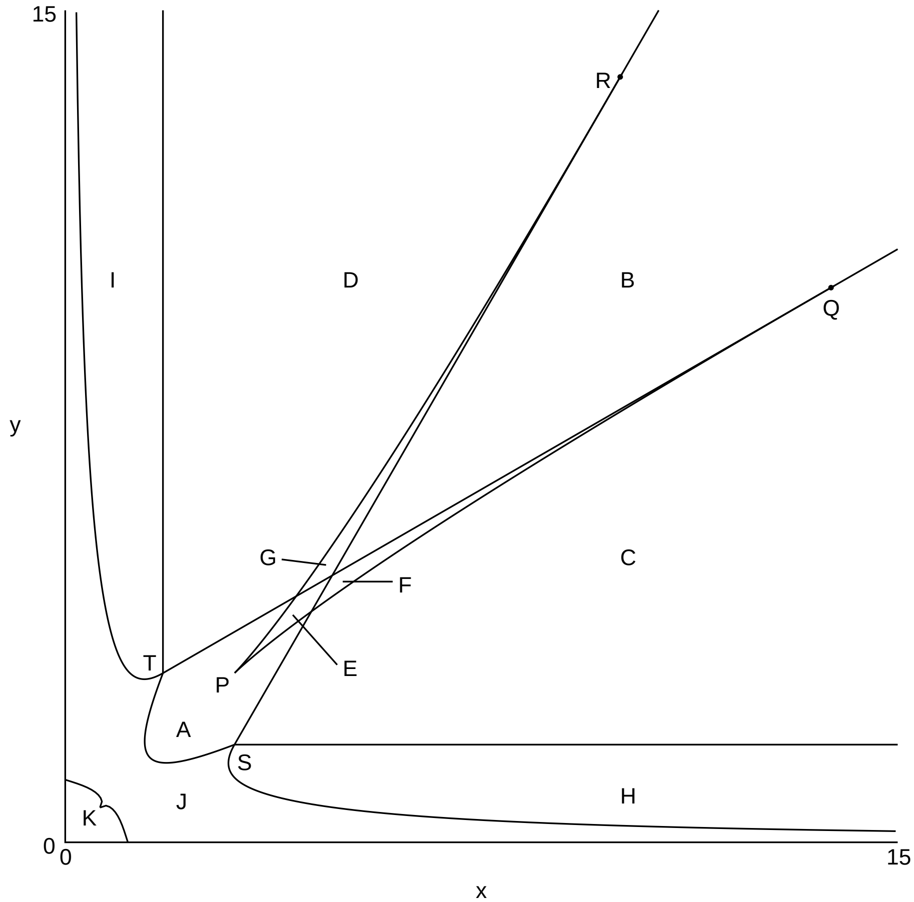
Substituting for r, s, t gives (5). Likewise for (6).

Figure 11 shows the transition between the X and the fat X with the arcs tangent at S. Since $x = r\cos t$ and $y = r\sin t$, one has $r = \sqrt{x^2 + y^2}$. The Plateau area is the triangle minus a quarter circle, so

$$A = \frac{1}{2}(2x)(2y) - \frac{\pi}{4}r^2, \tag{10}$$

whence (7).

Now we consider how the X's appear on figure 7. X films can be parameterized as shown in figure 12, which shows one quadrant. The arcs are tangent at point P, and line QR is perpendicular to the arcs at P. Segment PS is tangent to the arcs at P in equilibrium, making $\beta = t$, although to show instability we



Region A: Stable X.
Region B: Unstable X, stable horizontal and vertical double Y's.
Region C: Stable horizontal double Y.
Region D: Stable vertical double Y.
Region E: Stable horizontal and vertical X's, one unstable intermediate X.
Region F: One stable X, one unstable X, one stable horizontal double Y.
Region G: One stable X, one unstable X, one stable vertical double Y.
Region H: Stable fat horizontal double Y.
Region I: Stable fat vertical double Y.
Region J: Stable fat X.
Region K: Really fat nonunique absolute minima breaking rectangular symmetry, and other equilibria.
Curve PQ: Fold catastrophe line.
Curve PR: Fold catastrophe line.
Point P: Cusp catastrophe.

Figure 7: Stability diagram for rectangular X at constant area. The area is fixed at 1, while the coordinates of the pinned corner parameterize the diagram.

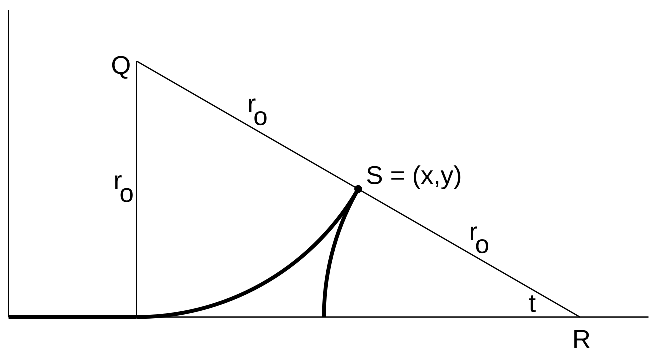


Figure 8: Transition between double Y and fat double Y. The arcs are tangent at S, and $t = \pi/6$.

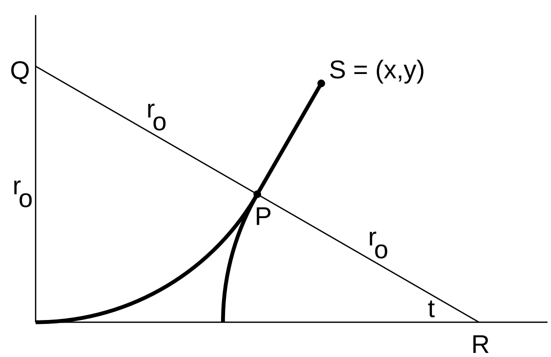


Figure 9: Transition between X and double Y. The arcs are tangent to the segment PS and $t = \pi/6$.

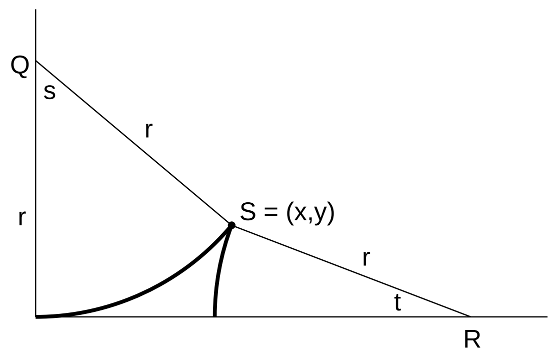


Figure 10: Transition between fat X and fat double Y. Note that angles s and t need not be complementary.

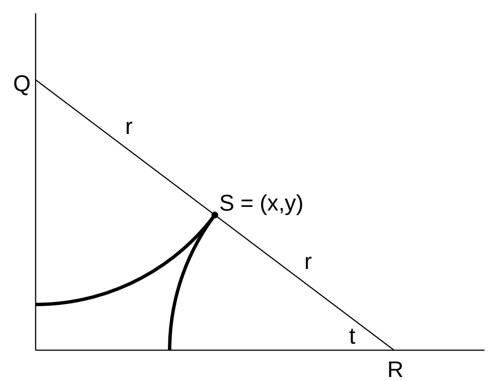


Figure 11: Transition between X and fat X. The arcs are tangent at S.

will later consider variations with $\beta \neq t$. Different values of (r, t) can correspond to the same (x, y), so that there can be more than one X with given corners. The Plateau area is

$$A(x, y, r, t) = r^2 \sin 2t - \frac{\pi}{4} r^2, \tag{11}$$

so if we fix $A = 1$ then

$$r(t) = (\sin 2t - \pi/4)^{-1/2}. \tag{12}$$

For equilibria, we have the condition

$$x - r \cos t = (y - r \sin t) \tan t. \tag{13}$$

Hence

$$r = \frac{x \cos t - y \sin t}{\cos^2 t - \sin^2 t}, \tag{14}$$

whence condition (8). The equilibrium surface in (x, y, t) space has a convenient parameterization in (u, t), where $u = PS$ and

$$x = u \sin t + \frac{\cos t}{\sqrt{\sin 2t - \pi/4}} \tag{15}$$

$$y = u \cos t + \frac{\sin t}{\sqrt{\sin 2t - \pi/4}} \tag{16}$$

for $\pi/6 \leq t \leq \pi/3$ and $u \geq 0$. When projected to (x, y), this surface has Jacobean

$$x_t y_u - x_u y_t = u - \frac{8 - 2\pi \sin 2t}{(4 \sin 2t - \pi)^{3/2}}. \tag{17}$$

Where the Jacobian is zero, there are fold lines in the projection. Here the fold lines reduce to the parametric curve

$$u = \frac{8 - 2\pi \sin 2t}{(4 \sin 2t - \pi)^{3/2}}, \qquad \pi/6 \leq t \leq \pi/3, \tag{18}$$

which is the curve QPR in figure 7. A short computation shows that this is also the curve $\partial A/\partial t = 0$ of case 5.

In regions A, C, D, H, I, and J stability follows from the uniqueness of the equilibrium. In other regions, the question of stability will be decided by reducing an arbitrary perturbation to a one-parameter perturbation, whose stability can be calculated explicitly. Since the film is defined as a map from the circle into the plane, the proper perturbation to consider is a perturbation of the map, rather than a perturbation of space. It will be shown that if there is any energy-reducing perturbation within a small neighborhood of the equilibrium, then there is an energy-reducing perturbation of a particular kind in a small neighborhood.

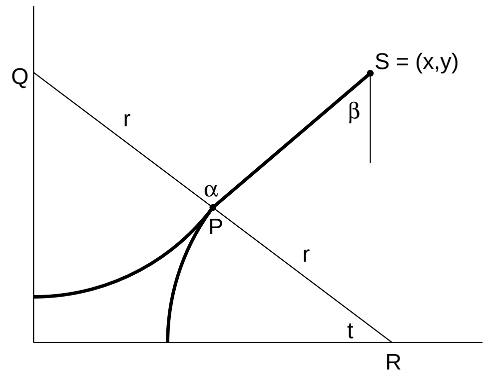


Figure 12: A quadrant of a perturbed X film. For equilibrium, $\alpha = \pi/2$ and $\beta = t$. More general nonequilibria (with $\alpha \neq \pi/2$) are used to decide the stability of certain equilibria.

The equilibria in the regions in question (B, E, F, G) are all X or double Y films. Note that if the corners are near enough to the triple points (regions A, C, D), the equilibrium is stable by uniqueness of the equilibrium, and the equilibrium is continuous in terms of the corner points.

If the equilibrium is unstable, then in an epsilon neighborhood of the film in the plane there will be a minimizing film of less energy. This film must have double-layered straight segments to near the original triple points, else staying in the epsilon neighborhood would force the curvature to be too small for the film to make it around the inner corners. On the straight segments, a rectangle of points can be found near the original triple points. To see this, start from a point on one segment and form the rectangular spiral starting at some angle and visiting the segments in turn, ending on the first segment. As the starting angle varies, by continuity the end point must match the starting point for some angle. The minimizing film with this rectangle as corner points is known, and is near the original film by continuity. We can reduce energy by straightening the double films between the corner points and the triple points. Fixing the shape of the Plateau border, force and torque balance show that the only equilibrium is the symmetric X or double Y, possibly with bends at the triple points. Hence symmetrizing lowers energy, and we have reduced a general perturbation to a symmetric bending at the triple points.

For the X film, such symmetric perturbations can be parameterized as shown in figure 12, which shows one quadrant. The arcs are tangent at point P, and line QR is perpendicular to the arcs at P. Segment PS is tangent to the arcs at P in equilibrium, making $\beta = t$, but not for the perturbation. The energy of the configuration is

$$E(x,y,r,t) = \frac{\pi}{2}r + 2\left[(x - r\cos t)^2 + (y - r\sin t)^2\right]^{1/2}. \qquad (19)$$

Calculating E_{tt} and substituting x, y from (15) and (16) gives

$$E_{tt} = \frac{(-4 + \pi\sin 2t)(-8 + 2\pi\sin 2t + u(4\sin 2t - \pi)^{3/2})}{4u(4\sin 2t - \pi)}. \qquad (20)$$

Solving for the only factor that can be zero gives

$$u_{crit} = \frac{8 - 2\pi\sin 2t}{(4\sin 2t - \pi)^{3/2}}, \qquad \pi/6 \le t \le \pi/3, \qquad (21)$$

which is exactly the equation for the fold line. Further, it is clear that the X is stable when $u > u_{crit}$ and unstable when $u < u_{crit}$, which gives the X film stabilities of the theorem. On the fold line $u = u_{crit}$, the film is unstable since the only other equilibrium is the absolutely minimizing double Y, which must have less energy. Likewise the limiting case double Y's of Case 5, in which opposite sides touch at one point, are unstable on the rays outward from Q and R.

A proper double Y film (opposite edges touching along a positive length) is always stable when it exists. Since the energy of a wet double Y differs from the energy of a dry double Y by a constant, and the dry double Y with straight segments is stable with respect to bending its straight segments, so is the wet double Y.

Finally we consider the double Y's on SR and TQ in which opposite sides touch at a point, which must be compared to nearby bent X's and bent Y's. The previous stability arguments for those two cases prove the asserted stability. Q.E.D.

In sum, where three equilibria exist, an unstable equilibrium is "between" two stable equilibria. Where a stable double Y occurs with other stable films, the absolute minimum appears to be the Y aligned with the long axis of the rectangle. We have not calculated this exactly due to the complexity of the formulas involved, but inspection of computer graphs of the energy indicate this.

The case of the wet square X is worthy of detailed description, translated to varying area in a fixed square:

Corollary 3.2 *Consider a wet film with corners fixed at* $(\pm 1, \pm 1)$ *and fixed area* $4A$. *Define the critical values* $A_1 = (\sqrt{3} - \pi/2)(2 - \sqrt{3}) \approx 0.04321$, $A_2 = (4 - \pi)/8 \approx 0.1073$, *and* $A_3 = (4 - \pi)/2 \approx 0.4292$.

If $0 < A \le A_1$, *then there are a stable horizontal double* Y, *an unstable symmetric* X, *and a stable vertical double* Y.

If $A_1 < A < A_2$, *then there are two asymmetric stable* X*'s (figure 2BC) and one unstable symmetric* X.

If $A_2 \leq A \leq A_3$, then there is one stable symmetric X.

If $A_3 < A < 1$, then liquid reaches the corners, and there is a stable symmetric fat X.

If $A \geq 1$, then many interesting things can happen, but that is beyond the scope of this paper.

4 Constant pressure.

If the wet X is a portion of a larger froth, then liquid can equilibrate throughout the froth, so it is more realistic to locally fix the external pressure difference $p > 0$ rather than area and minimize

$$E = L + pA, \tag{22}$$

where L is the film perimeter and A is the liquid area. Since the infimum of E is continuous in A, it occurs at some A_0. Conversely, any minimizer of E minimizes L for fixed A. Therefore existence and regularity follow from Theorem 2.1 and we consider the same equilibria. Indeed, $-p$ is just the Lagrange multiplier for the first problem of minimizing L for fixed A, p being equal to the curvature of the equilibrium.

Bolton and Weaire [BW,§3] treat this case, but misstate the number of equilibria in regions H, I, J of our figure 13 and the jump from H to A.

Theorem 4.1 *Consider a wet film with corners fixed at $(\pm x, \pm y)$ in equilibrium with external pressure $p = 1/r$. Then the following equilibria exist, with reference to figures 13 and 14:*

1. *A stable vertical double* Y *in regions B, D, and G:*

$$\frac{r}{2} \leq x < \sqrt{3}y - r. \tag{23}$$

2. *A stable horizontal double* Y *in regions B, C, and F:*

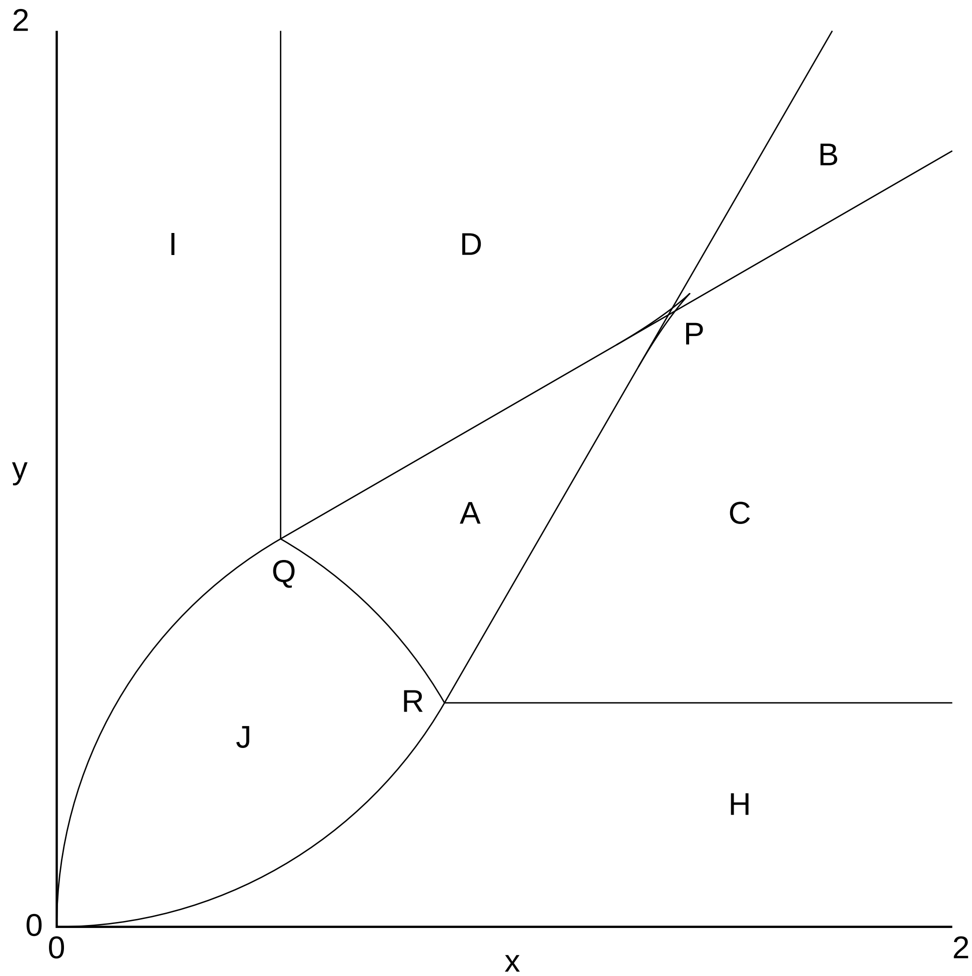
$$\frac{r}{2} \leq y < \sqrt{3}x - r. \tag{24}$$

3. *A stable horizontal fat double* Y *in region H:*

$$y < \frac{r}{2} \text{ and } x^2 + (y - r)^2 \geq r^2. \tag{25}$$

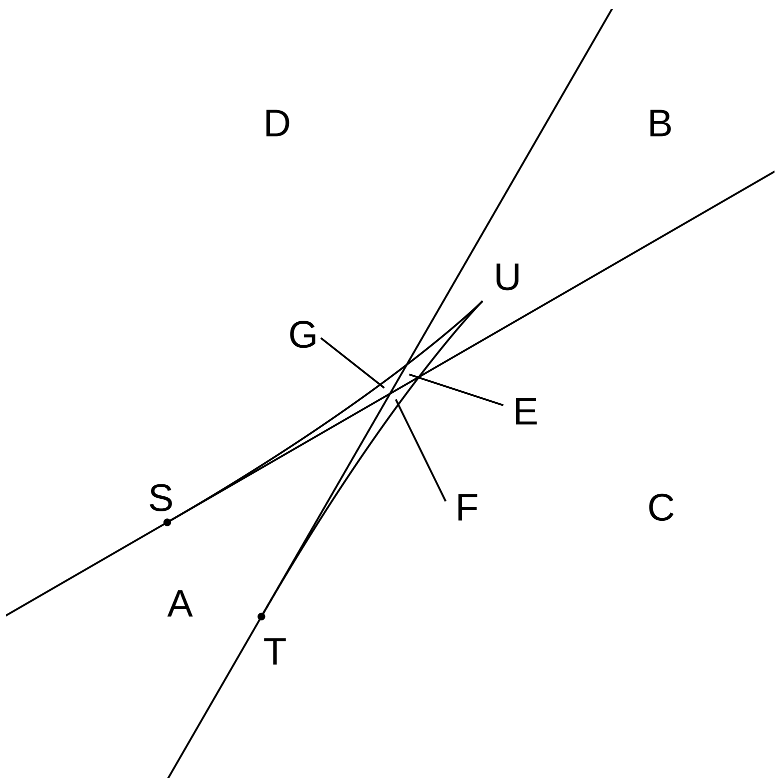
4. *A stable vertical fat double* Y *in region I:*

$$x < \frac{r}{2} \text{ and } (x - r)^2 + y^2 \geq r^2. \tag{26}$$



Region A: Stable X.
Region B: Unstable X, stable horizontal and vertical double Y's.
Region C: Stable horizontal double Y.
Region D: Stable vertical double Y.
Region H: Stable horizontal fat double Y.
Region I: Stable vertical fat double Y.
Region J: Fat X.

Figure 13: Stability diagram for rectangular X at constant pressure. The pressure is fixed at 1, while the coordinates of the pinned corner parameterize the diagram. The detail of the region near P is shown in figure 14.



Region A: Stable X.
Region B: Unstable X, stable horizontal and vertical double Y's.
Region C: Stable horizontal double Y.
Region D: Stable vertical double Y.
Region E: Two stable double Y's, unstable horizontal and vertical X's, stable intermediate X.
Region F: Stable horizontal double Y, unstable X, stable X.
Region G: Stable vertical double Y, unstable X, stable X.
Curve SUT: fold line.

Figure 14: Detail of the constant pressure stability diagram near point P of figure 13.

5. *An* X *in regions A, B, E, G, and F:*

$$x^2 + y^2 \geq r^2 \tag{27}$$

and there is some $\pi/6 < t < \pi/3$ *such that*

$$r = \frac{x \cos t - y \sin t}{\cos^2 t - \sin^2 t}. \tag{28}$$

There are one stable X *in region A, two stable* X*'s and one unstable* X *in region E, one stable* X *and one unstable* X *in regions F and G, and one unstable* X *in region B. The curves SU and TU are defined by* $\partial r/\partial t = 0$*, and indicate the merging of a stable* X *and an unstable* X *on one side of the curve into an unstable* X *on the curve. On rays RTP and QSP (*$t = \pi/6$ *and* $t = \pi/3$*, respectively), opposite sides meet at one point, and this double* Y *is stable on segments RP and QP, but unstable at S and T and beyond.*

6. *A fat* X *for region G:*

$$(x - r)^2 + y^2 < r^2, \qquad x^2 + (y - r)^2 < r^2, \qquad x^2 + y^2 < r^2. \tag{29}$$

Proof. The proof is similar to that of Theorem 3.1. The possible equilibria are as stated above and shown in figure 13. The interesting part is the location of the fold lines for the X films. Taking the external pressure to be 1 (so the arc radius is 1) and referring to figure 12, in (x, y, t) space the surface of equilibria is given by

$$x - \cos t = (y - \sin t) \tan t. \tag{30}$$

This can be conveniently parameterized in (u, t) as

$$x = \cos t + u \sin t \tag{31}$$

$$y = \sin t + u \cos t \tag{32}$$

for $\pi/6 \leq t \leq \pi/3$ and $u \geq 0$. On projection to (x, y) the fold line is

$$0 = x_u y_t - x_t y_u = \sin 2t - u. \tag{33}$$

So the fold line is just $u = \sin 2t$, which is the curve SRT in figure 14.

In regions A, C, D, H, I, and J, the equilibrium is unique and hence stable. The general stability of X and double Y films is handled as in the fixed area case.

The X stablity argument is the same as for fixed area, until the calculation of the stability of the bent symmetric X. The energy is

$$E = P + pA, \tag{34}$$

and, referring to figure 12, the energy of a quadrant is

$$E(x,y,t) = \frac{\pi}{2} + 2\left[(x-\cos t)^2 + (y-\sin t)^2\right]^{1/2} + 2\sin t\cos t - \frac{\pi}{4}. \tag{35}$$

The second derivative of energy on the equilibrium surface works out to be

$$E_{tt} = 2\sin 2t\left[\frac{\sin 2t}{u} - 1\right]. \tag{36}$$

Hence $E_{tt} = 0$ exactly on the fold line $u = \sin 2t$, and the film is stable for $u < \sin 2t$ and unstable for $u > \sin 2t$. For $u = \sin 2t$, it turns out that $E_{ttt} = 6\cos 2t$, so configurations on the fold line are unstable except at R.

A proper Y film is always stable, by an argument parallel to the fixed area case. Q.E.D.

As with fixed area, whenever a double Y exists along with other equilibria, the absolute minimum appears to be the double Y aligned with the rectangle.

The case of the square is interesting:

Corollary 4.2 *Consider a wet film with corners fixed at* $(\pm 1, \pm 1)$ *and fixed external pressure* p. *Define the critical values* $p_1 = 1/\sqrt{2} \approx 0.7071$, $p_2 = (\sqrt{3}+1)/2 \approx 1.366$, *and* $p_3 = \sqrt{2} \approx 1.414$.

If $0 < p < p_1$, *then there is a stable symmetric fat* X.

If $p_1 \leq p < p_2$, *then there is a stable symmetric* X.

If $p = p_2$, *then there are a stable symmetric* X *and two unstable double* Y*'s with opposite sides touching at one point.*

If $p_2 < p < p_3$, *then there are a stable vertical double* Y*, a stable horizontal double* Y*, a stable symmetric* X*, and two unstable asymmetric* X*'s.*

If $p \geq p_3$, *then there are a stable vertical double* Y*, a stable horizontal double* Y*, and an unstable symmetric* X.

5 Existence and regularity for unrestricted topological type

This section presents what current technology can prove about the existence and regularity of wet films of prescribed fluid area or volume and prescribed boundary but unrestricted topological type. Theorem 5.1 in $\mathbf{R}^2$ is much stronger than Theorem 5.2 in $\mathbf{R}^n$.

Theorem 5.1 *Among collections* $\{R_i : 1 \leq i \leq k\}$ *of disjoint planar regions (2-dimensional locally rectifiable currents of multiplicity 1 in* $\boldsymbol{R}^2$*) smoothly prescribed outside the unit ball and with*

$$0 < \sum_i area(R_i \cap (B(0,1)) < \pi \tag{37}$$

prescribed, there is a collection minimizing

$$\sum_i length(\partial R_i \cap B(0,1)). \tag{38}$$

Each $\partial R_i \cap B(0,1)$ is an embedded C^1 curve of finitely many components. For some $\kappa > 0$, the ∂R_i have constant curvature κ, except that they may coincide along straight line segments.

"Smoothly prescribed outside the unit ball" means there are smooth regions R_i^0 whose smooth boundaries intersect the unit circle transversally (in finitely many points) such that

$$spt(R_i - R_i^0) \subset B(0,1). \tag{39}$$

We may as well assume each R_i^0 intersects $B(0,1)$.

Proof. As in the proof of [M3, Thm 2.3], one obtains a minimizer and $\partial R_i \cap B(0,1)$ consists of finitely many simple Lipschitz curves, bounded only by the finitely many prescribed points on the unit circle. As in the proof of [M3, Thm. 3.2], these curves are C^1 on the interior of $B(0,1)$. A variational argument shows that each region with $area(R_i \cap B(0,1)) > 0$ has the same pressure p. Moreover $p > -1$ and hence every region has pressure p. The remaining regularity follows.

Remark. A similar result and proof hold for prescribing each $area(R_i \cap B(0,1))$ separately. Each $\partial R_i \cap B(0,1)$ is still an embedded, C^1, piecewise-constant-curvature curve of finitely many components, but the regions may have different pressures and bump up against each other and $\partial B(0,1)$ along circular arcs. One may further prescribe the combinatorial type of each region, but then multiple bumpings, degeneracies, and infinitely thin necks might occur in the minimizer, as in [M3, Thm. 3.2].

The known regularity in $\mathbf{R}^n$ is much weaker.

Theorem 5.2 *Among collections $\{R_i : 1 \leq i \leq k\}$ of disjoint regions in $\boldsymbol{R}^n$ (n-dimensional locally rectifiable currents of multiplicity 1 in $\boldsymbol{R}^n$) smoothly prescribed outside the unit ball and with prescribed total volume*

$$0 < \sum_i vol(R_i \cap B(0,1)) < vol(B(0,1)), \tag{40}$$

there is a collection minimizing

$$\sum_i area(\partial R_i \cap B(0,1)). \tag{41}$$

Each $\partial R_i \cap B(0,1)$ is an embedded C^1 manifold on an open dense subset of its support.

Proof. Existence follows from standard compactness arguments (see [M2]). The boundaries have weakly bounded mean curvature by an argument of Almgren ([A1, VI.2(3)], [M1, 13.5]; see [M1, §3]). Regularity follows by Allard's theorem [A, §8].

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